9.

Trigonometrical identities and equations

- The double angle identities
- $a \cos \theta + b \sin \theta$
- Sec θ, cosec θ and cot θ
- Product to sum and sum to product
- General solutions of trigonometric equations
- Obtaining the rule from the graph
- Modelling periodic motion
- Miscellaneous exercise nine

Situation

Three students, Jennifer, Ravi and Michael, were working on the same mathematics problem but obtained answers that appeared different.

	Jennifer's answer was	$1 - (2 \sin \theta)(\sin \theta - \cos \theta)$,
	Ravi's answer was	$2 \cos^2\theta - 1 + 2 \sin \theta \cos \theta$,
and	Michael's answer was	$\cos \theta + \sin \theta$.

Each of the students checked their working but did not discover any errors so were not prepared to admit to being wrong.

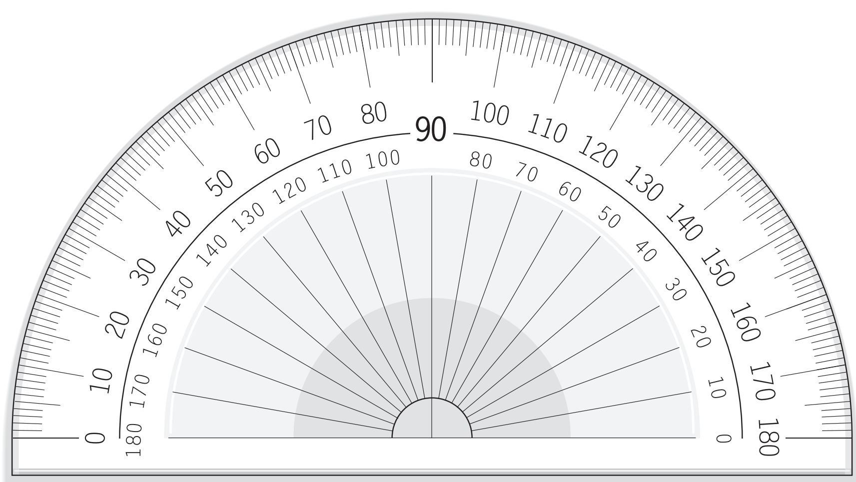
When they checked with the answers in the back of the book they found a different answer again! According to the book the answer was $\cos 2\theta + \sin 2\theta$.

Still unable to discover anything wrong with their own working they decided to evaluate each of the three answers for various values of θ.

- Evaluate Jennifer's answer for $\theta = 30°$.
- Evaluate Ravi's answer for $\theta = 30°$.
- Evaluate Michael's answer for $\theta = 30°$.
- Evaluate the book's answer for $\theta = 30°$.

Repeat this process for other values of θ.

Assuming that the answer in the back of the book was correct, did any of the three students also have the correct answer?



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The *Preliminary work* section for this unit reminded us of the fact that two apparently different trigonometrical expressions can actually be the same. In that case it was shown that the rule

$$y = -2\cos(x - 30°)$$

was the same as

$$y = 2\sin(x - 120°).$$

Similarly, in the situation on the previous page, whilst all four expressions that were involved may have appeared different, some were just different ways of writing the same thing.

The *Preliminary work* also reminded us of the Pythagorean **identity**:

$$\sin^2\theta + \cos^2\theta = 1$$

Remember that this is an *identity* because the left hand side equals the right hand side for **all** values of the variable. I.e. $\sin^2\theta + \cos^2\theta = 1$ for **all** values of θ.

Contrast this with the equation $2\sin\theta = 1$ which is only true for particular values of θ.

Note: In some texts the symbol ≡ is used for an identity.

For example $\sin^2\theta + \cos^2\theta \equiv 1$ is an identity,
but $2\sin\theta = 1$ is an equation.

The Pythagorean identity can be used to prove the truth of other identities, as shown in the next two examples.

EXAMPLE 1

Prove the identity: $\tan\theta - \sin\theta\cos\theta = \sin^2\theta\tan\theta$

Solution

$$\begin{aligned}\text{Left hand side} &= \tan\theta - \sin\theta\cos\theta \\ &= \frac{\sin\theta}{\cos\theta} - \sin\theta\cos\theta \\ &= \frac{\sin\theta - \sin\theta\cos^2\theta}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta}(1 - \cos^2\theta) \\ &= \tan\theta\sin^2\theta \\ &= \text{Right hand side}\end{aligned}$$

Note

The technique when proving an identity is to work on one side to see if it equals the other side.

Thus $\tan\theta - \sin\theta\cos\theta = \sin^2\theta\tan\theta$.

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Or, alternatively:

Right hand side = $\sin^2 \theta \tan \theta$

$$= (1 - \cos^2 \theta) \frac{\sin \theta}{\cos \theta}$$

$$= \frac{\sin \theta}{\cos \theta} - \cos^2 \theta \frac{\sin \theta}{\cos \theta}$$

$$= \tan \theta - \sin \theta \cos \theta$$

= Left hand side

Note

- If asked to prove an identity and you are not sure which side to start with then it is usually best to start with 'the more complicated side' and attempt to simplify it. This can be easier than starting with the 'simpler' side and knowing how it should be made more complicated.
- The above example involved $\tan \theta$ which is undefined for $\theta = 90°, 270°, \ldots$.
  When we say an identity is 'true for all values of θ' this really means 'for all values of θ for which the expression is defined'.

EXAMPLE 2

Prove the identity: $\tan \theta \cos \theta - \sin^3 \theta = \sin \theta \cos^2 \theta$

Solution

Left hand side = $\tan \theta \cos \theta - \sin^3 \theta$

$$= \frac{\sin \theta}{\cos \theta} \cos \theta - \sin^3 \theta$$

$$= \sin \theta - \sin^3 \theta$$

$$= \sin \theta (1 - \sin^2 \theta)$$

$$= \sin \theta \cos^2 \theta$$

= Right hand side

Note

The left hand side seems the more complicated.

Hence that is the side we will choose to start with.

Thus $\tan \theta \cos \theta - \sin^3 \theta = \sin \theta \cos^2 \theta$.

Exercise 9A

Prove the following identities.

1 $2\cos^2\theta + 3 = 5 - 2\sin^2\theta$

2 $\sin\theta - \cos^2\theta = (\sin\theta)(1 + \sin\theta) - 1$

3 $(\sin\theta + \cos\theta)^2 = 2\sin\theta\cos\theta + 1$

4 $1 - 2\sin\theta\cos\theta = (\sin\theta - \cos\theta)^2$

5 $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$

6 $\sin^4\theta - \sin^2\theta = \cos^4\theta - \cos^2\theta$

7 $\sin^2\theta\tan^2\theta = \tan^2\theta - \sin^2\theta$

8 $(1 + \sin\theta)(1 - \sin\theta) = 1 + (\cos\theta + 1)(\cos\theta - 1)$

9 $\sin\theta\tan\theta + \cos\theta = \dfrac{1}{\cos\theta}$

10 $\dfrac{1}{1+\tan^2\theta} = \cos^2\theta$

11 $\dfrac{\cos^2\theta + 2\cos\theta + 1}{\sin^2\theta} = \dfrac{1+\cos\theta}{1-\cos\theta}$

12 $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\sin\theta}$

13 $\dfrac{1-\sin\theta\cos\theta-\cos^2\theta}{\sin^2\theta+\sin\theta\cos\theta-1} = \tan\theta$

The *Preliminary work* reminded us of the angle sum and angle difference identities:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

These too can be used to prove various other identities, as the next example shows.

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EXAMPLE 3

Prove that $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{\tan A + \tan B}{\tan A - \tan B}$.

Solution

$$\begin{aligned}
\text{Left hand side} &= \frac{\sin(A+B)}{\sin(A-B)} \\
&= \frac{\sin A\cos B + \cos A\sin B}{\sin A\cos B - \cos A\sin B} \\
&= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\sin A\cos B}{\cos A\cos B} - \dfrac{\cos A\sin B}{\cos A\cos B}} \\
&= \frac{\tan A + \tan B}{\tan A - \tan B} \\
&= \text{Right hand side}
\end{aligned}$$

Thus $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{\tan A + \tan B}{\tan A - \tan B}$.

Exercise 9B

Prove the following identities.

1 $\sin(360° + \theta) = \sin\theta$

2 $\cos(360° + \theta) = \cos\theta$

3 $\sin(360° - \theta) = -\sin\theta$

4 $\cos(360° - \theta) = \cos\theta$

5 $\sin(A + B) - \sin(A - B) = 2\cos A\sin B$

6 $\cos(A - B) + \cos(A + B) = 2\cos A\cos B$

7 $2\cos\left(x - \dfrac{\pi}{6}\right) = \sin x + \sqrt{3}\cos x$

8 $\tan\left(\theta + \dfrac{\pi}{4}\right) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$

9 $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{1 - \tan A\tan B}{1 + \tan A\tan B}$

10 $\sqrt{2}(\sin x - \cos x)\sin(x + 45°) = 1 - 2\cos^2 x$

11 $\tan\left(\theta + \dfrac{\pi}{4}\right) = \dfrac{1 + 2\sin\theta\cos\theta}{1 - 2\sin^2\theta}$

The double angle identities

From the last section we know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Putting B equal to A we have $\sin(A+A) = \sin A\cos A + \cos A\sin A$

i.e. $\sin 2A = 2\sin A\cos A$

Similarly, from $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A+A) = \cos A\cos A - \sin A\sin A$

i.e. $\cos 2A = \cos^2 A - \sin^2 A$

$$\cos 2A = (1-\sin^2 A) - \sin^2 A = 1 - 2\sin^2 A$$

$$\cos 2A = \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$$

Thus:

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 1 - 2\sin^2 A\\ &= 2\cos^2 A - 1\end{aligned}$$

From $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ it follows that $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

EXAMPLE 4

Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Solution

$$\begin{aligned}\text{Left hand side} &= \sin 3\theta\\ &= \sin(\theta + 2\theta)\\ &= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta\\ &= \sin\theta(1-2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)\\ &= \sin\theta(1-2\sin^2\theta) + 2\sin\theta(1-\sin^2\theta)\\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\\ &= \text{Right hand side}\end{aligned}$$

Thus $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Note

(Although the left hand side seems less 'complicated' it involves $\sin 3\theta$, which we can attempt to break down into terms involving $\sin\theta$.)

The next examples show the use of the double angle identities in equation-solving.

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EXAMPLE 5

Solve $\sin 2x + \sin x = 0$ for $-\pi \le x \le \pi$.

Solution

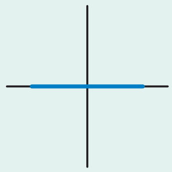
$$\sin 2x + \sin x = 0$$

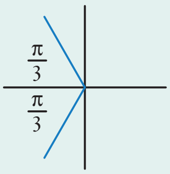
$\therefore$ $2 \sin x \cos x + \sin x = 0$

i.e. $\sin x(2 \cos x + 1) = 0$

$\therefore$ either $\sin x = 0$ or $2 \cos x + 1 = 0$

$\cos x = -0.5$





$x = -\pi, 0, \pi.$

$x = -\frac{2\pi}{3}, \frac{2\pi}{3}.$

Thus for $-\pi \le x \le \pi$ the solutions to $\sin 2x + \sin x = 0$ are $-\pi, -\frac{2\pi}{3}, 0, \frac{2\pi}{3}, \pi$.

EXAMPLE 6

a If $(2 \cos x + 1)(\cos x - 2) = a \cos^2 x + b \cos x + c$, determine a, b and c.

b Solve $\cos 2x = 3 \cos x + 1$ for $-180° \le x \le 180°$.

Solution

a Expanding: $(2 \cos x + 1)(\cos x - 2) = 2 \cos^2 x - 4 \cos x + \cos x - 2$

$= 2 \cos^2 x - 3 \cos x - 2$

Thus $a = 2$, $b = -3$ and $c = -2$.

b We are given the equation $\cos 2x = 3 \cos x + 1$

If we replace $\cos 2x$ by $(2 \cos^2 x - 1)$ we will obtain a quadratic in $\cos x$.

$\therefore$ $2 \cos^2 x - 1 = 3 \cos x + 1$

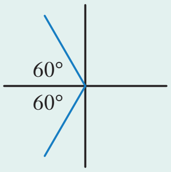
i.e. $2 \cos^2 x - 3 \cos x - 2 = 0$

hence, from **a** $(2 \cos x + 1)(\cos x - 2) = 0$

$\therefore$ either $2 \cos x + 1 = 0$ or $\cos x - 2 = 0$

i.e. $\cos x = -0.5$ or $\cos x = 2$

No solutions



$x = \pm 120°.$

Thus for $-180° \le x \le 180°$ the solutions to $\cos 2x = 3 \cos x + 1$ are $\pm 120°$.

Exercise 9C

1 If $\sin A = \frac{3}{5}$ and $90° \le A \le 180°$, find exact values for

a $\sin 2A$ **b** $\cos 2A$ **c** $\tan 2A$

2 If $\tan B = \frac{5}{12}$ and $\pi \le B \le \frac{3\pi}{2}$, find exact values for

a $\sin 2B$ **b** $\cos 2B$ **c** $\tan 2B$

3 Express each of the following in the form $a \sin bA$.

a $6 \sin A \cos A$ **b** $4 \sin 2A \cos 2A$ **c** $\sin \frac{A}{2} \cos \frac{A}{2}$

4 Express each of the following in the form $a \cos bA$.

a $2 \cos^2 2A - 2 \sin^2 2A$ **b** $1 - 2 \sin^2 \frac{A}{2}$ **c** $2 \cos^2 2A - 1$

5 If θ is obtuse and such that $\cos \theta = -\frac{24}{25}$ find exact values for

a $\sin 2\theta$ **b** $\cos 2\theta$ **c** $\tan 2\theta$

Solve the following equations for the given interval.

- With the occasional help of the information given in the display below right you should be able to solve these equations **without** the assistance of your calculator.
- Give exact answers where possible but if rounding is necessary give answers correct to one decimal place.

6 $4 \sin x \cos x = 1$ for $0 \le x \le 360°$.

7 $\sin 2x + \cos x = 0$ for $-180° \le x \le 180°$.

8 $2 \sin 2x - \sin x = 0$ for $0 \le x \le 360°$.

9 $2 \sin x \cos x = \cos 2x$ for $0 \le x \le 2\pi$.

10 $\cos 2x + 1 - \cos x = 0$ for $0 \le x \le 2\pi$.

11 $\cos 2x + \sin x = 0$ for $-\pi \le x \le \pi$.

12 $2 \sin^2 x + 5 \cos x + \cos 2x = 3$ for $0 \le x \le 540°$.

```
solve(cos(x)=0.25,x) |0 ≤x≤90°
                        {x=75.52248781}
solve(cos(x)=0.4,x) |0 ≤x≤90°
                        {x=66.42182152}
factor(2·y^2–y–1)
                        (2·y+1)·(y–1)
```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Prove the following identities.

13 $\sin 2\theta \tan \theta = 2 \sin^2 \theta$

14 $\cos \theta \sin 2\theta = 2 \sin \theta - 2 \sin^3 \theta$

15 $\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$

16 $\sin \theta \tan \dfrac{\theta}{2} = 2 - 2 \cos^2 \dfrac{\theta}{2}$

17 $\sin 4\theta = 4 \sin \theta \cos^3 \theta - 4 \sin^3 \theta \cos \theta$

18 $\dfrac{\sin 2\theta - \sin \theta}{1 - \cos \theta + \cos 2\theta} = \tan \theta$

19 $\cos 4\theta = 1 - 8 \cos^2 \theta + 8 \cos^4 \theta$

a cos θ + *b* sin θ

Consider the function $y = 3 \cos \theta + 4 \sin \theta$

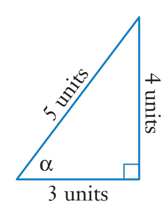
Writing this as $y = 5\left(\dfrac{3}{5}\cos \theta + \dfrac{4}{5}\sin \theta\right)$

(The 5 being chosen because $\sqrt{3^2 + 4^2} = 5$.)

it follows that $y = 5 (\cos \theta \cos \alpha + \sin \theta \sin \alpha)$
$= 5 \cos (\theta - \alpha)$

where $\cos \alpha = \dfrac{3}{5}$

and $\sin \alpha = \dfrac{4}{5}$



as shown in the diagram on the right.

Using a calculator $\alpha = 53.1°$ (correct to 1 decimal place).

Thus the initial expression $y = 3 \cos \theta + 4 \sin \theta$

can be rearranged to $y = 5 \cos (\theta - 53.1°)$

This means that the graph of $y = 3 \cos x + 4 \sin x$ will be that of $y = \cos x$ but with an amplitude 5 and moved right 53.1°. Confirm this to be the case by viewing the graph of $y = 3 \cos x + 4 \sin x$ on a calculator or computer graphing package.

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EXAMPLE 7

Express 2 cos θ – 3 sin θ in the form *a* cos (θ + α) for known values of *a* (exact) and α an acute angle (in degrees correct to one decimal place).

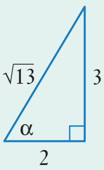
Solution

We rearrange to the form cos *A* cos *B* – sin *B* sin *A* using $\sqrt{2^2+3^2}$, i.e. $\sqrt{13}$.

$$2\cos\theta - 3\sin\theta = \sqrt{13}\left(\frac{2}{\sqrt{13}}\cos\theta - \frac{3}{\sqrt{13}}\sin\theta\right)$$

$$= \sqrt{13}(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

$$= \sqrt{13}\cos(\theta + \alpha)$$



Using a calculator, α = 56.3° (correct to one decimal place)

Thus $\quad 2\cos\theta - 3\sin\theta = \sqrt{13}\cos(\theta + 56.3°)$

This is of the required form with $a = \sqrt{13}$ and α = 56.3°.

Again check that the graph of *y* = 2 cos *x* – 3 sin *x* is indeed that of *y* = cos *x* stretched vertically, scale factor $\sqrt{13}$, and with an appropriate phase shift.

EXAMPLE 8

a Express 5 sin θ + 12 cos θ in the form *R* sin (θ + α) for α an acute angle in degrees, correct to one decimal place.

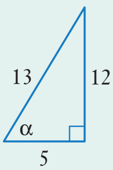
b Hence determine the maximum value of 5 sin θ + 12 cos θ and the smallest positive value of θ for which it occurs (correct to one decimal place).

Solution

a $$5\sin\theta + 12\cos\theta = \sqrt{5^2+12^2}\left(\frac{5}{\sqrt{5^2+12^2}}\sin\theta + \frac{12}{\sqrt{5^2+12^2}}\cos\theta\right)$$

$$= 13\left(\frac{5}{13}\sin\theta + \frac{12}{13}\cos\theta\right)$$

$$= 13(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$



Using a calculator we find that α = 67.4° (correct to 1 decimal place).

Hence $\quad 5\sin\theta + 12\cos\theta = 13\sin(\theta + 67.4°)$.

b sin (θ + 67.4°) has a maximum value of 1 when (θ + 67.4°) = 90°

Thus 5 sin θ + 12 cos θ has a maximum value of 13 when θ = 90° – 67.4°

= 22.6°

In example 7, we expressed 2 cos θ – 3 sin θ in the form $\sqrt{13}$ cos (θ + 56.3°). This rearrangement can be useful if we were asked to solve an equation of the form 2 cos θ – 3 sin θ = *c*, as the next example shows.

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EXAMPLE 9

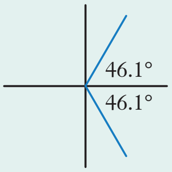
Use the rearrangement of example 7 to solve 2 cos x – 3 sin x = 2.5 for $0 \le x \le 360°$.

Solution

Given: $2\cos x - 3\sin x = 2.5$

Hence: $\sqrt{13}\cos(x + 56.3°) = 2.5$

$$\cos(x + 56.3°) = \frac{5}{2\sqrt{13}}$$



$$(x + 56.3°) = \ldots, 46.1°, 313.9°, 406.1°, \ldots .$$

$$x = \ldots, -10.2°, 257.6°, 349.8°, \ldots .$$

Thus for $0 \le x \le 360°$ the solutions to $2\cos x - 3\sin x = 2.5$ are 257.6° and 349.8°.

Exercise 9D

Express each of the following in the form $a\cos(\theta + \alpha)$ for α an acute angle in degrees correct to one decimal place.

1 $3\cos\theta - 4\sin\theta$

2 $12\cos\theta - 5\sin\theta$

Express each of the following in the form $a\cos(\theta - \alpha)$ for α an acute angle in radians correct to two decimal places.

3 $4\cos\theta + 3\sin\theta$

4 $7\cos\theta + 24\sin\theta$

Express each of the following in the form $a\sin(\theta + \alpha)$ for α an acute angle in degrees correct to one decimal place.

5 $5\sin\theta + 12\cos\theta$

6 $7\sin\theta + 24\cos\theta$

Express each of the following in the form $a\sin(\theta - \alpha)$ for α an acute angle in radians correct to two decimal places.

7 $4\sin\theta - 3\cos\theta$

8 $2\sin\theta - 3\cos\theta$

9 Use your answers to questions 7 and 8 to sketch the graphs of $y = 4\sin x - 3\cos x$ and $y = 2\sin x - 3\cos x$. (Then check your sketches using a graphic calculator.)

10 **a** Express $\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$ for α an acute angle in radians.

b Hence determine the maximum value of $\cos\theta + \sin\theta$ and the smallest positive value of θ for which it occurs. (Give θ in radians.)

11 Solve $3\cos x + 4\sin x = 2$ for $0 \le x \le 2\pi$. (Answers correct to 2 decimal places.)

12 Solve $10\sin x + 5\cos x = 8$ for $-\pi \le x \le \pi$. (Answers correct to 2 decimal places.)

13 Solve $2\sin x + 5\cos x = 3$ for $0 \le x \le 2\pi$. (Answers correct to 2 decimal places.)

Sec θ, cosec θ and cot θ

WS Advanced exponential and periodic functions

The reciprocals of sin θ, cos θ and tan θ, i.e. $\frac{1}{\sin\theta}$, $\frac{1}{\cos\theta}$ and $\frac{1}{\tan\theta}$, can occur frequently and are given names of their own:

$$\frac{1}{\cos\theta} = \sec\theta \qquad \frac{1}{\sin\theta} = \operatorname{cosec}\theta \qquad \frac{1}{\tan\theta} = \cot\theta \left(= \frac{\cos\theta}{\sin\theta}\right)$$

Note:
- Sec, cosec and cot are abbreviations for secant, cosecant and cotangent.
- We prefer to define cot θ as $\frac{\cos\theta}{\sin\theta}$, then $\cot\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = 0$,

  thus avoiding $\cot\left(\frac{\pi}{2}\right) = \frac{1}{\tan\left(\frac{\pi}{2}\right)} \left(= \frac{1}{\text{undefined}}\right)$.

Pythagorean identities can be established for these reciprocal functions as follows:

$$\sin^2\theta + \cos^2\theta = 1$$

Divide by $\cos^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

i.e.

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

Divide by $\sin^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

i.e.

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

These Pythagorean identities can be used as they are (see example 12) or the question can be re-written in terms of sine, cosine and tangent and then solved as before (see examples 10 and 11).

EXAMPLE 10

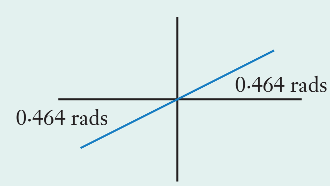
Solve $\cot x = 2$ for $-\pi \le x \le \pi$.

Solution

If $\quad \cot x = 2$

then $\quad \frac{1}{\tan x} = 2$

and so $\quad \tan x = \frac{1}{2}$



Thus for $-\pi \le x \le \pi$ the solutions of $\cot x = 2$ are 0.46 rads and –2.68 rads, correct to 2 decimal places.

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EXAMPLE 11

Prove the identity: $\sec\theta - \cos\theta = \sin\theta\tan\theta$

Solution

$$\begin{aligned}\text{Left hand side} &= \sec\theta - \cos\theta \\ &= \frac{1}{\cos\theta} - \cos\theta \\ &= \frac{1-\cos^2\theta}{\cos\theta} \\ &= \frac{\sin^2\theta}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \times \sin\theta \\ &= \tan\theta\sin\theta \\ &= \text{Right hand side}\end{aligned}$$

Thus $\sec\theta - \cos\theta = \sin\theta\tan\theta$.

EXAMPLE 12

Solve $8\cot^2\theta = 14\operatorname{cosec}\theta - 13$ for $-180° \le \theta \le 180°$.

Solution

$$8\cot^2\theta = 14\operatorname{cosec}\theta - 13$$

By substituting for $8\cot^2\theta$, we can obtain a quadratic in $\operatorname{cosec}\theta$:

$$8(\operatorname{cosec}^2\theta - 1) = 14\operatorname{cosec}\theta - 13$$

$$8\operatorname{cosec}^2\theta - 14\operatorname{cosec}\theta + 5 = 0$$

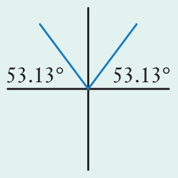
$$(4\operatorname{cosec}\theta - 5)(2\operatorname{cosec}\theta - 1) = 0$$

Hence either $4\operatorname{cosec}\theta - 5 = 0$ or $2\operatorname{cosec}\theta - 1 = 0$

$\operatorname{cosec}\theta = \frac{5}{4}$ or $\operatorname{cosec}\theta = \frac{1}{2}$

$\sin\theta = \frac{4}{5}$ or $\sin\theta = 2$

no solutions



$\theta = 53.13°, 126.87°$

Thus for $-180° \le \theta \le 180°$ the solutions to $8\cot^2\theta = 14\operatorname{cosec}\theta - 13$ are 53.13° and 126.87°.

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Exercise 9E

Solve the following equations for the given interval.

1 $\sec x = 2$ for $0 \le x \le 2\pi$.

2 $3\operatorname{cosec}^2 x = 4$ for $-\pi \le x \le \pi$.

3 $\sin x \sec x - 3\sin x = 0$ for $0 \le x \le 360°$.

4 $(\sec x)(3 - \sec x) = \tan^2 x - 1$ for $-180° \le x \le 180°$.

5 $5\cos x = \sec x$ for $0 \le x \le 360°$.

6 $\operatorname{cosec}\left(x + \frac{\pi}{3}\right) = \sqrt{2}$ for $0 \le x \le 2\pi$.

7 $\sec^2 x + \sec x = 2$ for $0 \le x \le 360°$.

8 $2\cot^2 x + 5\operatorname{cosec} x - 1 = 0$ for $0 \le x \le 2\pi$.

Prove the following identities.

9 $1 = \sin^2\theta\cot^2\theta + \sin^2\theta$

10 $(\cot^2\theta)(1 - \cos^2\theta) = 1 - \sin^2\theta$

11 $1 + \cot^2\theta = \cot^2\theta\sec^2\theta$

12 $(\sec\theta - 1)(\operatorname{cosec}\theta + \cot\theta) = \tan\theta$

13 $\tan^4\theta - 1 = \tan^2\theta\sec^2\theta - \sec^2\theta$

14 $\frac{1+\sin\theta}{1-\sin\theta} = 2\tan^2\theta + 1 + 2\tan\theta\sec\theta$

15 $\frac{1+\sin\theta}{1-\sin\theta} = \sec^2\theta + 2\tan\theta\sec\theta + \tan^2\theta$

16 $\frac{1+\sec\theta}{1-\sec\theta} = 1 - 2\operatorname{cosec}^2\theta - 2\cot\theta\operatorname{cosec}\theta$

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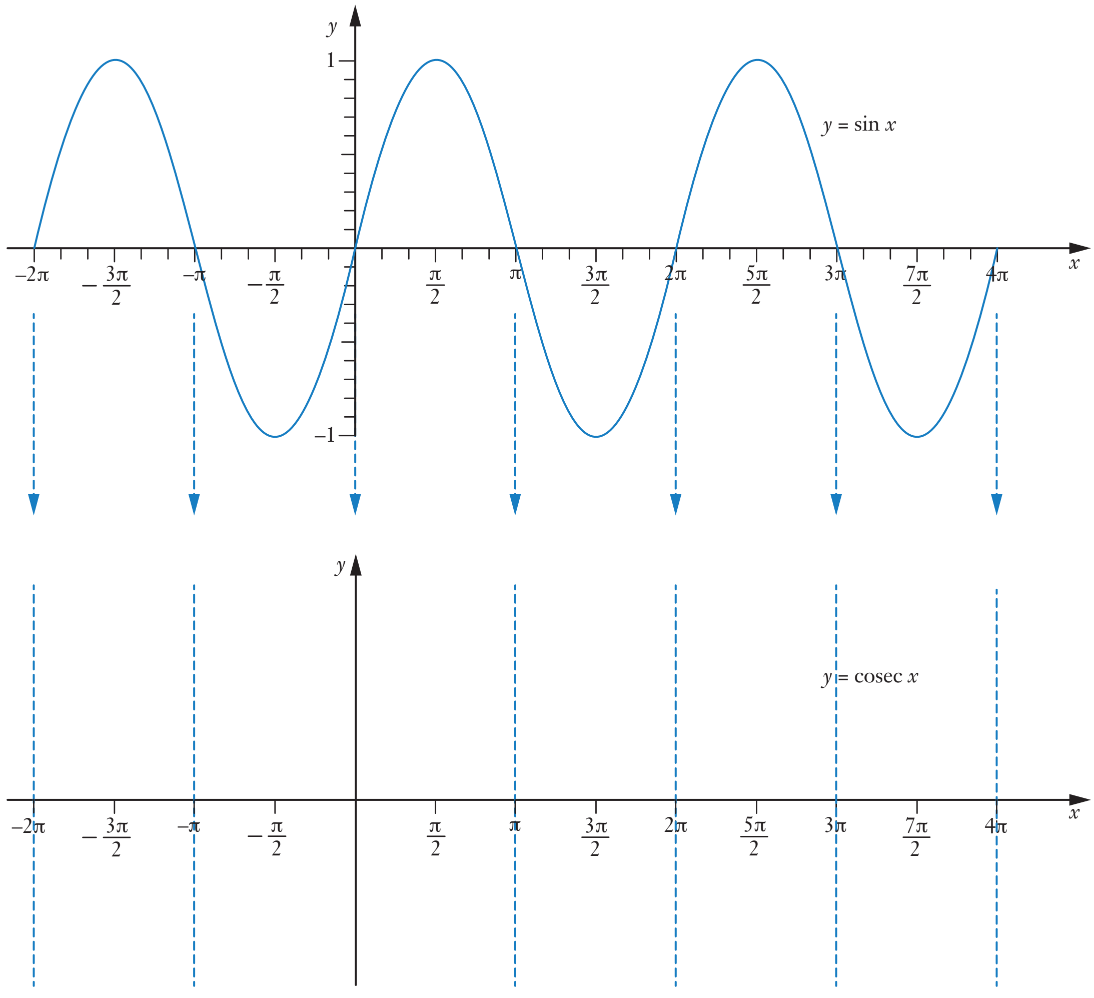
17 This question requires you to think about what the graph of $y = \text{cosec}\ x$ looks like.

The graph below shows $y = \sin x$ for $-2\pi \le x \le 4\pi$, which involves 3 cycles.

With cosec x being $\frac{1}{\sin x}$ it follows that the graph of $y = \text{cosec}\ x$ will be asymptotic for any values of x for which $\sin x = 0$.

Make a copy of the diagram shown below and try to sketch the graph of $y = \text{cosec}\ x$ for $-2\pi \le x \le 4\pi$ on the lower set of axes.



Then view the graph of $y = \text{cosec}\ x$ on a graphic calculator or computer graphing package and see how your sketch compares.

Similarly produce sketches of $y = \sec x$ and $y = \cot x$ for $2\pi \le x \le 4\pi$ and then check the correctness of your sketches by comparing them to the display from a graphic calculator or computer.

How will the graphs of

$y = a\ \text{cosec}\ x$, $\quad y = \text{cosec}\ bx$, $\quad y = \text{cosec}\ (x - c)$, $\quad y = d + \text{cosec}\ x$

compare with that of $y = \text{cosec}\ x$? Investigate.

Product to sum and sum to product

Given $\sin(A+B) = \sin A\cos B + \cos A\sin B$ [1]

and $\sin(A-B) = \sin A\cos B - \cos A\sin B$ [2]

[1] + [2] $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

writing $A + B = P$ and $A - B = Q$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \qquad \sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

The identity above right is sometimes remembered as

'sine + sine equals 2 sine semi-sum, cos semi-difference'.

[1] – [2] $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

writing $A + B = P$ and $A - B = Q$

$$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \qquad \sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Given $\cos(A+B) = \cos A\cos B - \sin A\sin B$ [3]

and $\cos(A-B) = \cos A\cos B + \sin A\sin B$ [4]

[3] + [4] $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

writing $A + B = P$ and $A - B = Q$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)] \qquad \cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

[3] – [4] $\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

writing $A + B = P$ and $A - B = Q$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \qquad \cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

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Summary:

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$	$\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$	$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$
$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$	$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$	$\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

The identities on the left can be useful if we wish to express the products

'cos cos', 'sin sin', 'sin cos', 'cos sin'

as sums or differences.

The identities on the right can be useful if we wish to express the sums or differences

'cos ± cos', 'sin ± sin'

as products.

EXAMPLE 13

Solve $4 \sin 5x \sin 3x + 2 \cos 8x = 1$ for $0 \le x \le 360°$.

Solution

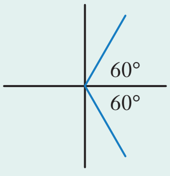
Given: $4 \sin 5x \sin 3x + 2 \cos 8x = 1$

$\therefore$ $4 \times \frac{1}{2}[\cos(5x - 3x) - \cos(5x + 3x)] + 2\cos 8x = 1$

$2\cos 2x - 2\cos 8x + 2\cos 8x = 1$

$2\cos 2x = 1$

$\cos 2x = 0.5$



$2x = 60°, 300°, 420°, 660°$

$x = 30°, 150°, 210°, 330°$

Solutions are 30°, 150°, 210°, 330°.

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EXAMPLE 14

Prove that $\dfrac{\sin 7x - \sin 3x}{\cos 6x + \cos 4x} = 2 \sin x$.

Solution

$$\text{Left hand side} = \frac{\sin 7x - \sin 3x}{\cos 6x + \cos 4x}$$

$$= \frac{2\cos\left(\frac{7x+3x}{2}\right)\sin\left(\frac{7x-3x}{2}\right)}{2\cos\left(\frac{6x+4x}{2}\right)\cos\left(\frac{6x-4x}{2}\right)}$$

$$= \frac{2\cos 5x \sin 2x}{2\cos 5x \cos x}$$

$$= \frac{\sin 2x}{\cos x}$$

$$= \frac{2\sin x \cos x}{\cos x}$$

$$= 2 \sin x$$

$= \text{Right hand side}$

Exercise 9F

Express each of the following as the sum or difference of two 'trig' functions.

1 $\cos 3x \cos 2x$ **2** $\sin 3x \sin x$ **3** $\sin 7x \cos x$ **4** $\cos 3x \sin x$

Express each of the following as the product of two 'trig' functions.

5 $\cos 5x + \cos x$ **6** $\cos 5x - \cos x$ **7** $\sin 6x + \sin 2x$ **8** $\sin 5x - \sin 3x$

9 Express the product $\sin 75° \cos 15°$ as an exact value.

10 Express $\sin 75° + \sin 15°$ as an exact value.

11 Solve: $4 \sin 7x \cos 2x = \sqrt{3} + 2 \sin 9x$ for $0 \le x \le 180°$.

12 Solve: $\sin 7x + \sin 3x = \sin 5x$ for $0 \le x \le \pi$.

13 Solve: $\sin 3x - \sin x = 0$ for $0 \le x \le 360°$

14 Solve: $\sin 5x \cos 3x = \sin 6x \cos 2x$ for $-\pi \le x \le \pi$.

15 Prove that: $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$.

16 Prove that $\sqrt{2}\cos\left(2x - \dfrac{\pi}{4}\right) - \dfrac{\sin 7x + \sin 3x}{2 \sin 5x} = \sin 2x$.

17 Prove that: $\cos 8A \cos 2A - \cos 7A \cos 3A + \sin 5A \sin A = 0$.

18 Prove that: $4 \sin 3A \sin 2A \cos A = 1 + \cos 2A - \cos 4A - \cos 6A$.

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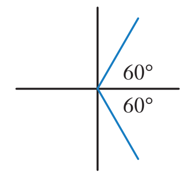
General solutions of trigonometric equations

In all of the trigonometric equations we have been asked to solve so far in this text, the solutions we have been required to find have been restricted to some given interval. Suppose instead we were asked to find **all** of the solutions to a particular trigonometric equation?

When asked to solve the equation

$$\cos x = 0.5 \qquad \text{for } 0 \le x \le 360°$$



our knowledge of exact values and 'what's positive where' allows us to create the drawing shown on the right, and state the solutions in the required interval as

$$x = 60°, 300°.$$

Given a different interval, say $-180° \le x \le 180°$ our diagram again allows the solutions to be determined: $x = \pm 60°$.

Asked to determine **all** solutions to the equation $\cos x = 0.5$ there will be an infinite number! However all of these solutions will be as shown in the diagram, but with some number of complete rotations added or subtracted.

Thus the general solution of the equation $\cos x = 0.5$ is

$$x = \begin{cases} 60° + n \times 360°, \\ -60° + n \times 360°, \end{cases} \quad \text{for } n \in \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Writing $\mathbb{Z}$ for the set of integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ this general solution can be

written $\quad x = \begin{cases} 60° + n \times 360°, \\ -60° + n \times 360°, \end{cases} \quad \text{for } n \in \mathbb{Z},$

or as $\quad x = n \times 360° \pm 60°$ for $n \in \mathbb{Z}$.

In this way the expression

$x = n \times 360° \pm 60°$ for $n \in \mathbb{Z}$, gives **all** solutions to the equation $\cos x = 0.5$.

$n = \ldots$

$n = -3$	gives	$x = -1140°, -1020°$
$n = -2$	gives	$x = -780°, -660°$
$n = -1$	gives	$x = -420°, -300°$
$n = 0$	gives	$x = -60°, 60°$
$n = 1$	gives	$x = 300°, 420°$
$n = 2$	gives	$x = 660°, 780°$
$n = 3$	gives	$x = 1020°, 1140°$

$n = \ldots$

$x = n \times 360° \pm 60°$ for $n \in \mathbb{Z}$ is referred to as the **general solution** of the equation.

Does your calculator give general solutions to trigonometric equations? Investigate.

On the previous page, we used the solutions of –60° and +60° to obtain:

$$x = \begin{cases} 60° + n \times 360°, \\ -60° + n \times 360°, \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

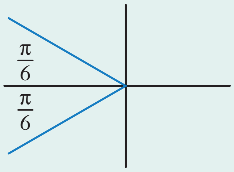
Had we instead used 60° and 300° we would have written

$$x = \begin{cases} 60° + n \times 360°, \\ 300° + n \times 360°, \end{cases} \quad \text{for } n \in \mathbb{Z},$$

Do these two general statements generate the same solutions? Investigate.

EXAMPLE 15

Find all solutions to the equation $\sqrt{3} + 2 \cos x = 0$, for x in radians.



Solution

If $\sqrt{3} + 2 \cos x = 0$

$$2 \cos x = -\sqrt{3}$$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$\text{Thus } x = \begin{cases} \frac{5\pi}{6} + n \times 2\pi, \\ \frac{7\pi}{6} + n \times 2\pi, \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

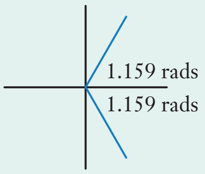
(Alternatively this could also be written as $x = 2n\pi \pm \frac{5\pi}{6}$.)

EXAMPLE 16

Find all solutions to the equation $\cos(3(x - 2)) = 0.4$, giving answers correct to 2 decimal places.

Solution

(With no indication to the contrary, we assume radians to be the unit of angle measure.)
Using a calculator to obtain the acute angle for which $\cos x = 0.4$, and an awareness of 'what's positive where' allows us to create the diagram on the right.



$$\text{Thus} \quad 3(x - 2) = \begin{cases} 1.159 + n \times 2\pi, \\ -1.159 + n \times 2\pi, \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

$$x - 2 = \begin{cases} 0.386 + \frac{2n\pi}{3}, \\ -0.386 + \frac{2n\pi}{3}, \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

$$x = \begin{cases} 2.39 + \frac{2n\pi}{3}, \\ 1.61 + \frac{2n\pi}{3}, \end{cases} \quad \text{for } n \in \mathbb{Z}, \text{ correct to two decimal places.}$$

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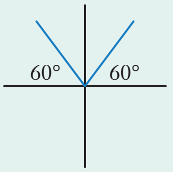
EXAMPLE 17

Find all solutions to the equation $\sin x = \frac{\sqrt{3}}{2}$, for x in degrees.

Solution



$$\sin x = \frac{\sqrt{3}}{2}$$

$$x = \begin{cases} 60° + n \times 360°, \\ 120° + n \times 360°, \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

Alternatively, if we consider the solutions to be generated by

adding 60° to even multiples of 180°

and subtracting 60° from odd multiples of 180°,

the general solution could be written as

$$x = \begin{cases} 2n \times 180° + 60°, \\ (2n+1) \times 180° - 60°, \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

(which could then be simplified to give the earlier version.)

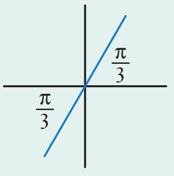
EXAMPLE 18

Find all solutions to the equation $\tan x = \sqrt{3}$.

Solution

(With no indication to the contrary, we assume radians to be the unit of angle measure.)

$$\tan x = \sqrt{3}$$



$$x = \begin{cases} \frac{\pi}{3} + 2n\pi, \\ \frac{4\pi}{3} + 2n\pi, \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

Alternatively, if we consider the solutions to be generated by adding $\frac{\pi}{3}$ to each multiple of π, the general solution could be written as

$$x = n\pi + \frac{\pi}{3} \text{ for } n \in \mathbb{Z}.$$

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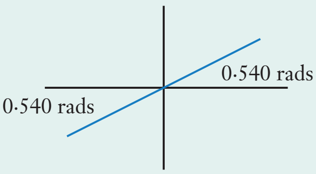
EXAMPLE 19

Find all solutions to the equation $\tan\left(\frac{\pi}{2}(2x-1)\right) = 0.6$, giving answers correct to 2 decimal places.

Solution

(With no indication to the contrary, we assume radians to be the unit of angle measure.)

Using a calculator to obtain the acute angle for which $\tan x = 0.6$, and an awareness of 'what's positive where' allows us to create the diagram on the right.



$$\frac{\pi}{2}(2x - 1) = n\pi + 0.540 \text{ for } n \in \mathbb{Z}.$$

$$2x - 1 = \frac{2}{\pi}(n\pi + 0.54) \text{ for } n \in \mathbb{Z}.$$

$$= 2n + 0.344 \text{ for } n \in \mathbb{Z}.$$

$$2x = 2n + 1.344 \text{ for } n \in \mathbb{Z}.$$

$$x = n + 0.67 \text{ for } n \in \mathbb{Z}, \text{ correct to 2 decimal places.}$$

Exercise 9G

Find all solutions to each of the following equations for x in degrees.

(Give answers correct to 1 decimal place if rounding is necessary.)

1 $\sin x = 0.5$

2 $\cos x = 1$

3 $\tan x = -\frac{1}{\sqrt{3}}$

4 $\sin(2x + 30°) = 1$

5 $\cos(3(x - 20°)) = 0.7$

6 $\tan(2(x + 10°)) = 0.8$

Find all solutions to each of the following equations.

(Give answers correct to 2 decimal places if rounding is necessary and, with no indication to the contrary, assume radians to be the unit of angle measure.)

7 $4 \sin x \cos x = -1$

8 $\sin^3 x + \sin x \cos^2 x = \cos x$

9 $\cos^2 x - \sin^2 x = 1$

10 $\sin 2x \cos x + \cos 2x \sin x = 0.5$

11 $\cos(4(x - 1)) = 0.8$

12 $2 \sin 3x \sin x + \cos 4x = 0.5$

13 $\cos\left(3x - \frac{\pi}{4}\right) = 0$

14 $\sin\left(\frac{\pi}{4}(3x - 1)\right) = 0.25$

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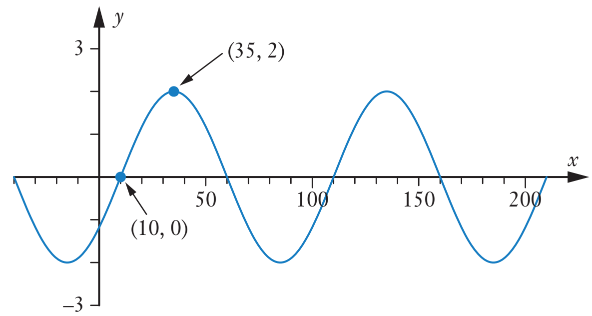
Obtaining the rule from the graph

The graph on the right looks like it could be the graph of $y = \sin x$ that has been

- moved right ten units
- stretched parallel to the y-axis, scale factor 2, and
- dilated parallel to x axis.



This would suggest that the given graph has an equation of the form $y = 2 \sin [b(x - 10)]$.

Now $y = \sin bx$ performs b cycles in 2π units of x-axis. (Using radians, not degrees).

I.e. $y = \sin bx$ has a period of $\frac{2\pi}{b}$.

The given graph has a period of 100, thus $100 = \frac{2\pi}{b}$

giving $b = \frac{\pi}{50}$

Hence the graph shown has equation $\quad y = 2 \sin \left(\frac{\pi}{50}(x - 10) \right)$

Display such a function on a graphic calculator or computer graphing package to see if the rule does indeed give the graph shown above.

Of course, we could instead have considered the given graph to be that of $y = \cos x$ moved right 35 units, stretched parallel to the y-axis, and dilated horizontally. This would suggest the rule of the graph would be $y = 2 \cos \left(\frac{\pi}{50}(x - 35) \right)$.

However we already know that trigonometric expressions that appear different can sometimes be just different ways of expressing the same thing so the fact that there are two 'different' rules for the graph shown above should be no great surprise (and was also mentioned in the earlier *Preliminary work* section for this unit).

Use the fact that $\sin x = \cos \left(\frac{\pi}{2} - x \right)$ to show the equivalence of

$$y = 2 \sin \left(\frac{\pi}{50}(x - 10) \right) \qquad \text{and} \qquad y = 2 \cos \left(\frac{\pi}{50}(x - 35) \right)$$

Indeed, why stop at just two versions? Check that each of the following rules also give the graph shown at the top of this page:

$$y = 2 \sin \left(\frac{\pi}{50}(x + 90) \right) \qquad y = 2 \sin \left(\frac{\pi}{50}(x - 110) \right) \qquad y = -2 \sin \left(\frac{\pi}{50}(x + 40) \right)$$

Hence if asked to determine the rule for a given trigonometric graph do not be too quick to assume your answer is wrong even if it does not look quite the same as the answer obtained by someone else.

WS Sketching periodic functions—amplitude and period

WS Sketching periodic functions—phase and vertical shift

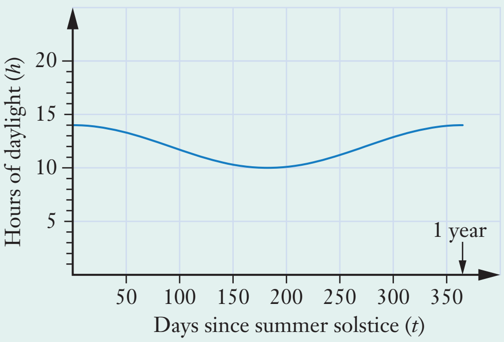
WS Sketching periodic functions

Modelling periodic motion

Some real-life situations follow periodical patterns in their variation that can be quite well modelled by an appropriate trigonometrical relationship.

EXAMPLE 20

Let us suppose that in a particular part of the world the number of hours of daylight (h hours) can be reasonably well modelled by the graph shown on the right, with the graph commencing at the summer solstice (the longest day) and showing one year (365 days).



Determine an equation for the graph in the form $h = a\cos(bt) + d$.

Solution

The graph seems to have a high value for h of 14 and a low of 10. Thus the mean value of h is 12, hence $d = 12$, and the amplitude is 2, i.e. $a = 2$.

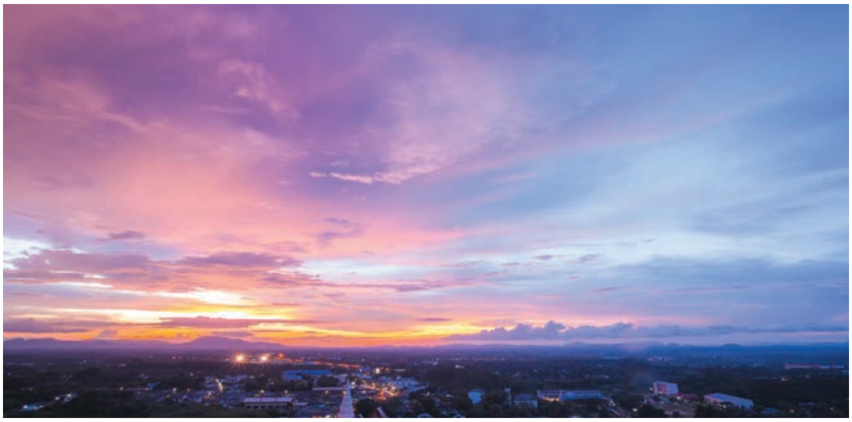
The period of $y = \cos bx$ is $\frac{2\pi}{b}$ and the given graph has a period of 365.

Thus $\quad \frac{2\pi}{b} = 365$

Hence $\quad b = \frac{2\pi}{365}$

The required equation is $\quad h = 2\cos\left(\frac{2\pi}{365}t\right) + 12$

The reader should use a graphic calculator or computer graphing program to check that the above rule does indeed give the graph shown above.



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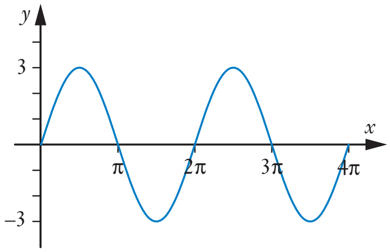
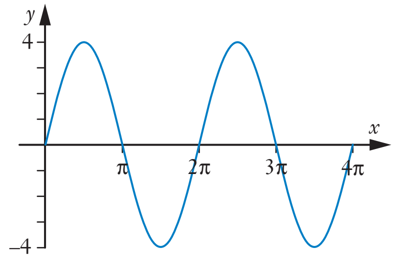
Exercise 9H

1 Write the equation of each of the following in the form $y = a \sin x$.

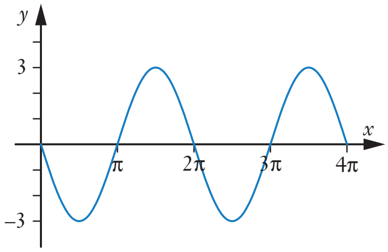
a



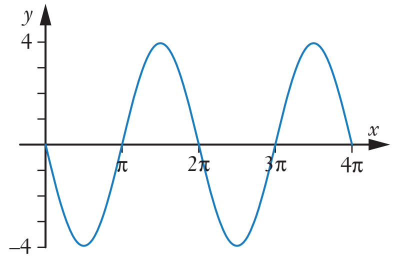
b



c

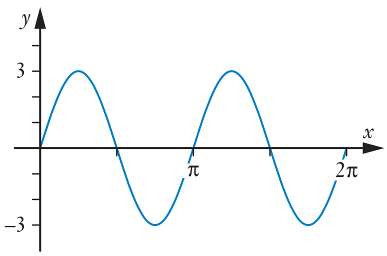


d

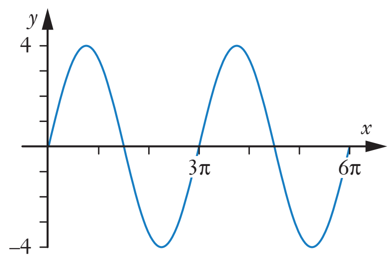


2 Write the equation of each of the following in the form $y = a \sin bx$.

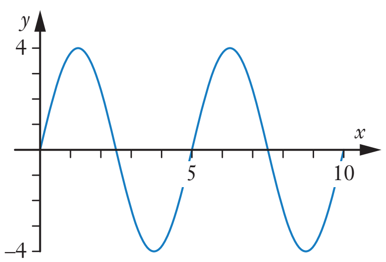
a



b



c



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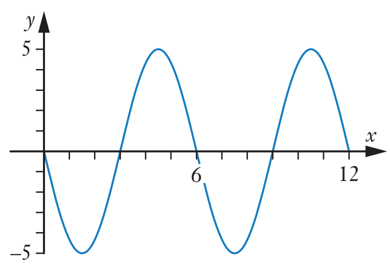
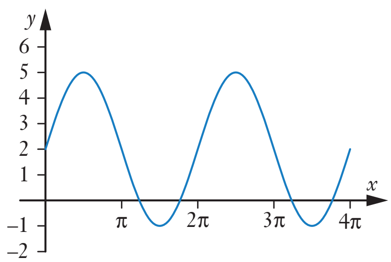
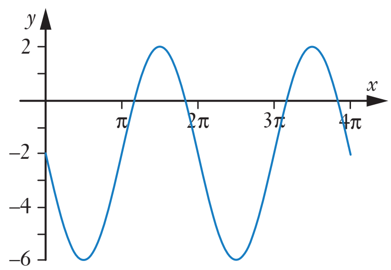


3 Write the equation of each of the following in the form $y = d + a \sin x$.

a



b



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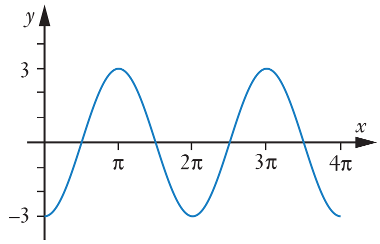
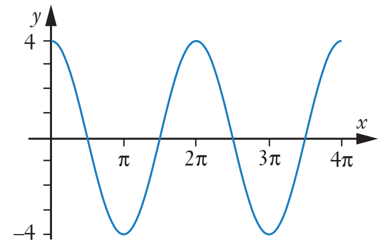
4 Write the equation of each of the following in the form $y = a \sin(x + c)$.

a

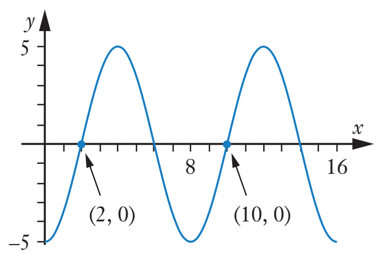


b

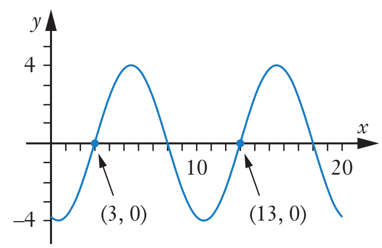


5 Write the equation of each of the following in the form $y = a \sin[b(x + c)]$.

a

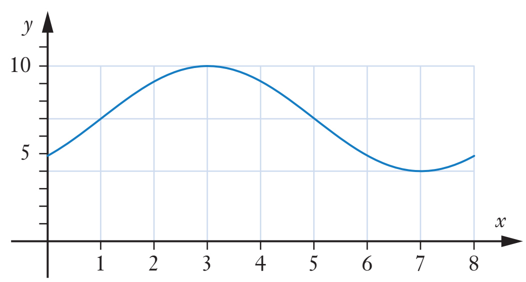


b

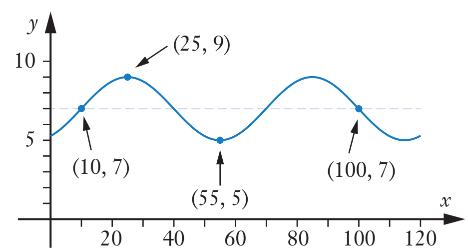


6 Write the equation of each of the following in the form $y = a \sin[b(x + c)] + d$.

a

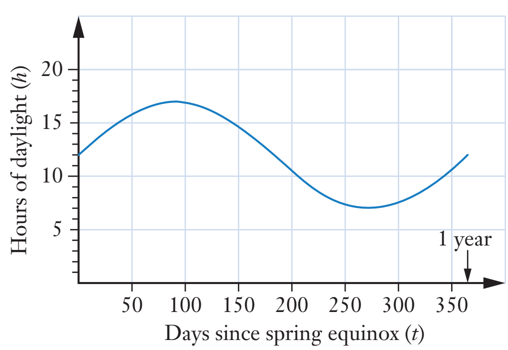


b



7 Let us suppose that in a particular part of the world the number of hours of daylight (h hours) can be reasonably well modelled by the graph shown on the right, with the graph commencing at the spring equinox (day and night of equal length) and showing one year (365 days).

Determine an equation for the graph in the form $h = a \sin(bt) + d$, given that the longest day had 17 hours of daylight and the shortest day had just 7 hours of daylight.



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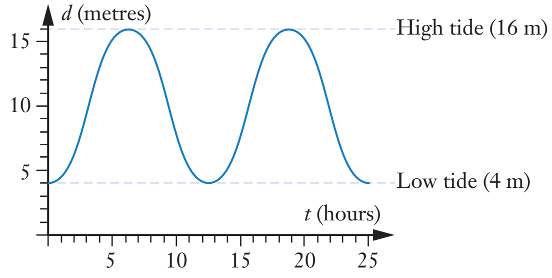
8 An automatic device records the depth of water (d metres) in a tidal harbour from one low tide until two low tides later, a time interval of 25 hours. It was found that d plotted against t, the number of hours since recording commenced, fitted closely to the graph shown below.

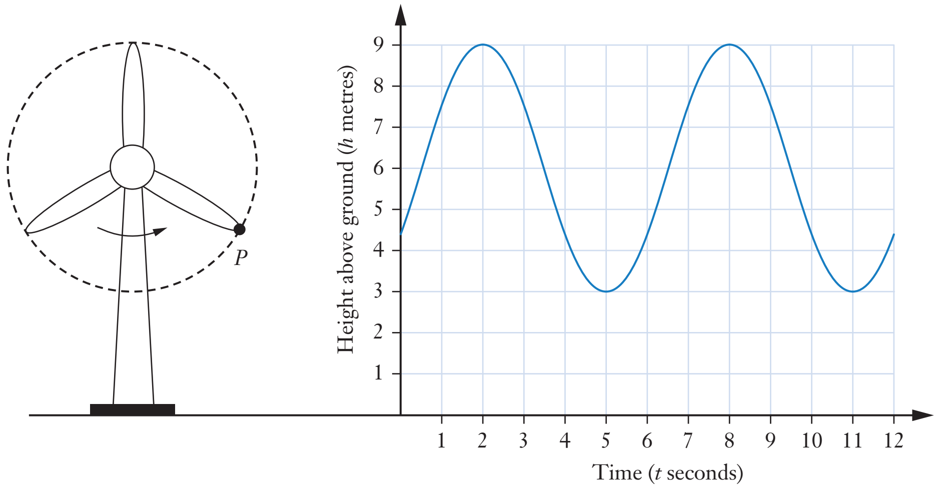
a Determine an equation for the graph in the form $d = a \cos(bt) + e$.



b Express the relationship between d and t as a sine function.

9 The graph below right shows the height above ground (h metres) of the tip of one of the blades of a wind farm turbine at time t seconds.

a Express the relationship between h and t in the form $h = a \cos[b(t - c)] + d$.



b Express the relationship between h and t as a sine function.

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Miscellaneous exercise nine

This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the Preliminary work section at the beginning of this unit.

1 Find all solutions to the equation $\sqrt{2}\sin 5x = 1$ lying in the interval $0 \le x \le \pi$.

2 Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

3 Solve $2\sin x\cos x = \sqrt{3} - 2\sqrt{3}\sin^2 x$ for $0 \le x \le 360°$.

4 Solve $\tan^2 x = 3(\sec x - 1)$ for $-\pi \le x \le \pi$.

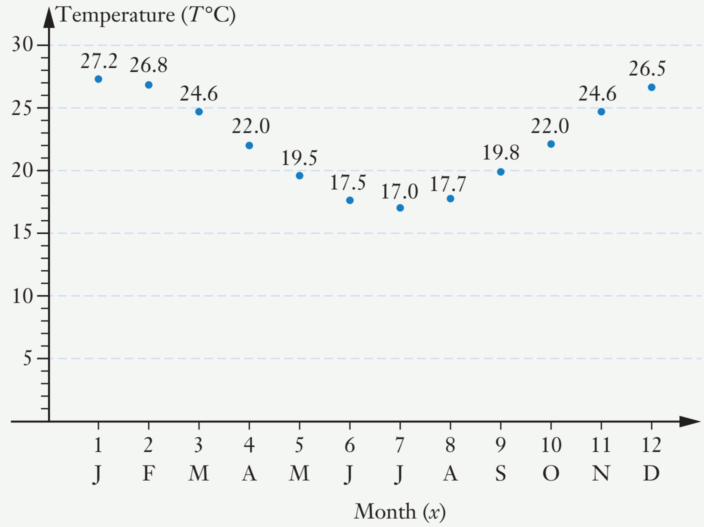
5 Find all solutions to the equation $4\sin 3x\cos x = \sqrt{3} + 2\sin 2x$ for x in radians.

6 **a** Express $7\sin\theta - 10\cos\theta$ in the form $R\sin(\theta - \alpha)$ for α an acute angle in radians, correct to two decimal places and R exact.

b Hence determine the minimum value of $7\sin\theta - 10\cos\theta$ and the smallest positive value of θ for which it occurs. (Give θ in radians.)

7 The mean monthly daily temperatures for a particular location were as shown tabulated and graphed below:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
27.2	26.8	24.6	22.0	19.5	17.5	17.0	17.7	19.8	22.0	24.6	26.5



a *Without* using the ability of some calculators to determine a sinusoidal model to fit given data points, obtain an equation of the form $T = a\sin[b(x + c)] + d$ to model the above situation, explaining your reasoning.

b If you have a calculator or computer program that will find a sinusoidal model for given data points, use it to determine a suitable model and compare it to your answer for part **a**.

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